UNIFORMLY BOUNDED REPRESENTATIONS AND EXACT GROUPS

KATE JUSCHENKO AND PIOTR W. NOWAK

ABSTRACT. We characterize groups with Guoliang Yu's property A (i.e., exact groups) by the existence of a family of uniformly bounded representations which approximate the trivial representation.

Property A is a large scale geometric property that can be viewed as a weak counterpart of amenability. It was shown in [12], that for a finitely generated group property A implies the Novikov conjecture. It was also quickly realized that this notion has many other applications and interesting connections, see [9, 10].

A well-known characterization of amenability states that the constant function 1 on G, as a coefficient of the trivial representation, can be approximated by diagonal, finitely supported coefficients of the left regular representation of G on $\ell_2(G)$. In this note we prove a counterpart of this result for groups with property A in terms of uniformly bounded representations. A representation π of a group G on a Hilbert space H is said to be uniformly bounded if $\sup_{g\in G}\|\pi_g\|_{B(H)} < \infty$.

Theorem 1. *Let G be a finitely generated group equipped with a word length function. G has property A (i.e., G is exact) if and only if for every $\varepsilon > 0$ there exists a uniformly bounded representation π of G on a Hilbert space H, a vector $v \in H$ and a constant $S > 0$ such that*

(1) $\|\pi_g v\| = 1$ *for all* $g \in G$,
(2) $|1 - \langle \pi_g v, \pi_h v\rangle| \le \varepsilon$ *if* $|g^{-1}h| \le 1$,
(3) $\langle \pi_g v, \pi_h v\rangle = 0$ *if* $|g^{-1}h| \ge S$.

Alternatively, the second condition can be replaced by an almost-invariance condition: $\|\pi_g v - \pi_h v\| \le \varepsilon$ if $|g^{-1}h| \le 1$. Another characterization of property A in this spirit, involving convergence for isometric representations on Hilbert C^*-modules was studied in [4].

Recall that the Fell topology on the unitary dual is defined using convergence of coefficients of unitary representations. Theorem 1 states that the trivial representation can be approximated by uniformly bounded representations, in a fashion similar to Fell's topology.

Similar phenomena were considered by M. Cowling [2, 3] in the case of the Lie group $\mathrm{Sp}(n,1)$. Recall that $\mathrm{Sp}(n,1)$ has property (T), and thus the trivial representation is an isolated point among the equivalence classes of unitary

Date: 22 April 2013.

representations in the Fell topology. Cowling showed that nevertheless, for $\mathrm{Sp}(n,1)$ the trivial representation can be approximated by uniformly bounded representations in a certain sense. Theorem 1 gives a similar statement for all discrete groups with property A. Recall that almost all known groups with property (T) are known to have property A. In particular, the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, satisfy property A [5].

Moreover, under a stronger assumption that the group has Hilbert space compression strictly greater than 1/2 in the sense of [6], we obtain a path of uniformly bounded representations, whose coefficients continuously interpolate between the trivial and the left regular representation.

Theorem 1 suggests the possibility of negating property A using strengthened forms of Kazhdan's property that applies to uniformly bounded representations.

Question 1. *Are there finitely generated groups satisfying a sufficiently strong version of property (T) for uniformly bounded representations, so that these groups cannot have property A?*

Certain versions of such a property (T) for uniformly bounded representations were considered by Cowling [2, 3], but they would not apply directly in our case. Construction of new examples of finitely generated groups without property A is a major open problem in coarse geometry, with possible applications in operator algebras, index theory and topology of manifolds.

Acknowledgements. We would like to thank Guoliang Yu for helpful comments as well as Pierre-Nicolas Jolissaint, Mikael de la Salle and Alain Valette for carefully reading the first version of the paper and suggesting several significant improvements.

1. Uniformly bounded representations and property A

Let H_0 be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_0$, and let T be a bounded, positive, self-adjoint operator on H_0. We additionally assume that T has a spectral gap; that is, there exists $\lambda > 0$ such that

$$\langle v, Tv\rangle_0 \geq \lambda \langle v, v\rangle_0 \tag{1}$$

for every $v \in H_0$.

The operator T induces a new inner product on V, the vector space underlying H_0, by the formula

$$\langle v, w\rangle_T = \langle v, Tw\rangle_0.$$

The norm $\|v\|_T$ induced by $\langle\cdot,\cdot\rangle_T$ is equivalent to the original norm on H_0, since

$$\lambda\|v\|_0^2 \leq \|v\|_T^2 \leq \|T\|_{B(H_0)}\, \|v\|_0^2.$$

Thus we obtain a new Hilbert space H_T by equipping V with the norm induced by T. A unitary representation π on H_0 naturally becomes a uniformly

bounded representation on H_T. More precisely, the norm of the representation satisfies

$$\|\pi\| = \sup_{g\in G} \|\pi_g\|_{B(H_T)} \leq \frac{\|T\|_{B(H_0)}}{\lambda}.$$

In the Hilbert space H_T, the representation π satisfies

$$\pi_g^* = T^{-1}\pi_{g^{-1}}T,$$

for every $g \in G$.

We will now relate property A to the existence of uniformly bounded representations with the desired properties. From the discussion on renormings via positive operators we derive the following fact.

Lemma 2. *Let G be a finitely generated group equipped with a word length function and $\varepsilon > 0$. If there exists a Hilbert space H_0, a positive self-adjoint bounded operator T of H satisfying (1) , a unitary representation π, a unit vector $v \in H_0$ and $S > 0$ such that*

1. *$\langle \pi_g v, T\pi_g v\rangle_H = 1$ for every $g \in G$,*
2. *$|1 - \langle \pi_g v, T\pi_h v\rangle_H| \leq \theta$ whenever $|g^{-1}h| \leq 1$,*
3. *$\langle \pi_g v, T\pi_h v\rangle_H = 0$ whenever $|g^{-1}h| \geq S$,*

then there exists a uniformly bounded representation π of G on a Hilbert space H_T and $v \in H_T$, satisfying the conditions listed in Theorem 1.

Proof. Let V denote the vector space underlying H. We equip V with a scalar product $\langle v, w\rangle_T = \langle v, Tw\rangle_0$ and obtain the space H_T as explained in the previous section. Viewing π and v with respect to this new norm gives the required properties. □

Recall that a Hermitian kernel on a set X is a function $K : X \times X \to \mathbb{C}$ such that $K(x, y) = \overline{K(y, x)}$. K is said to be positive definite if for every finitely supported function $f : X \to \mathbb{C}$ we have

$$\sum_{x,y\in X} K(x, y)f(x)f(y) \geq 0.$$

Positive definite kernels can be used to characterize property A, we use this description as the definition. We refer to [9–11] for more details and other characterizations of property A.

Theorem 3 (see [11])**.** *A discrete metric space X has property A if and only if for every $R, \varepsilon > 0$ there exists a Hermitian positive definite kernel $K : X \times X \to [0, 1]$ and $S > 0$, satisfying*

1. *$K(x, x) = 1$ for every $x \in X$,*
2. *$|1 - K(x, y)| \leq \varepsilon$ if $d(x, y) \leq R$,*
3. *$K(x, y) = 0$ if $d(x, y) \geq S$.*

For a finitely generated group G we take X to be G with the word length metric. In that case it suffices to consider only $R = 1$. A Hermitian kernel K on X induces a self-adjoint linear operator on $\ell_2(X)$, denoted also by K, by

viewing K as a matrix over X. We will identify the operator with the kernel representing it.

Lemma 4. *Let G be a finitely generated group with Yu's property A. Then for every $\varepsilon > 0$ there exists an operator T of a Hilbert space H, a unitary representation π of G on H and a unit vector $v \in H$, satisfying the conditions of lemma 2.*

Proof. Let $\varepsilon > 0$. Given K as in Theorem 3, define an operator

$$T = \frac{1}{1+\varepsilon}(K + \varepsilon I),$$

where I is the identity on H.

It is clear that since K is a positive operator, T is also positive. It is easy to check that since T is represented by a kernel, which takes values in the interval $[0,1]$ and vanishes outside of a neighborhood of the diagonal, T is a bounded operator on $\ell_2(G)$. Finally,

$$\langle v, Tv\rangle = \langle v, Kv\rangle + \varepsilon\langle v, v\rangle \geq \varepsilon\langle v, v\rangle.$$

Thus T is a positive, self-adjoint, bounded operator of $H_0 = \ell_2(G)$ and it satisfies (1). Consequently we can construct a new Hilbert space H_T, isomorphic to $\ell_2(G)$, as explained earlier.

Consider now π, the left regular representation of G on $\ell_2(G)$, viewed as a representation on H_T. By the previous discussion, π is a uniformly bounded representation on H_T.

Denote by δ_g the Dirac mass at $g \in G$ and let $v = \delta_e$. Whenever $g \neq h$ we have

$$\langle \pi_g v, T\pi_h v\rangle = \frac{1}{1+\varepsilon}\langle \pi_g v, K\pi_h v\rangle = \frac{1}{1+\varepsilon}\langle \delta_g, K\delta_h\rangle = \frac{1}{1+\varepsilon}K(g,h), \tag{2}$$

and

$$\|\pi_g v\|_T = \langle \delta_g, T\delta_g\rangle = 1.$$

For $g, h \in G$ such that $|g^{-1}h| = 1$ we can estimate

$$\begin{aligned} |1 - \langle \pi_g v, T\pi_h v\rangle| &= |1 - \langle \delta_g, T\delta_h\rangle| \\ &= |1 - T(g,h)| \\ &= |1 - \frac{1}{1+\varepsilon}K(g,h)| \\ &\leq \varepsilon + \frac{\varepsilon}{1+\varepsilon}, \end{aligned}$$

by (2). Also,

$$\langle \pi_g v, T\pi_h v\rangle = \frac{1}{1+\varepsilon}\langle \pi_g v, K\pi_h v\rangle = 0,$$

whenever $|g^{-1}h| \geq S$. Thus T, π and v satisfy the required conditions with S and $\varepsilon' = \varepsilon + \dfrac{\varepsilon}{1+\varepsilon} \leq 2\varepsilon$. ◻

We are now in the position to prove the main theorem.

Proof of Theorem 1. If G is a finitely generated group with property A then we apply lemma 4 and lemma 2 and the claim follows.

Conversely, given $\varepsilon > 0$, the corresponding representation π and a vector v define $K(g,h) = \langle \pi_g v, \pi_h v\rangle$. Then K is positive definite and it is easy to check that it satisfies the conditions required by Theorem 3. □

A path of representations. Let G be a finitely generated group. G coarsely embeds into the Hilbert space H if there exists a map $f : G \to H$, two non-decreasing functions $\rho_-, \rho_+ : [0,\infty) \to [0,\infty)$ such that

$$\rho_-(d(g,h)) \leq \|f(g) - f(h)\|_H \leq \rho_+(d(g,h)),$$

and $\lim_{t\to\infty} \rho_-(t) = \infty$. Such an f is called a coarse embedding.

It is shown in [6] that if there exists $\theta > 0$ such that $\rho_-(t) \geq Ct^{1/2+\theta} + D$ for $t \geq E$, for some constants $C, D, E > 0$, then the positive definite kernel

$$K_\alpha(g,h) = e^{-\alpha\|f(g)-f(h)\|_H^2},$$

induces a bounded positive operator on $\ell_2(G)$. The proof relies on the Schur test. The existence of θ as above is strictly stronger than property A. Similarly as above we can use these kernels to construct uniformly bounded representations.

Let $f : [0,\infty) \to [0,\infty)$ be a smooth function such that

(1) $\lim_{t\to 0} f(\alpha) = 0$,
(2) $\lim_{t\to\infty} f(\alpha)$ exists.

Applying the previous construction to the operators

$$T_\alpha = K_\alpha + f(\alpha)I,$$

we obtain a family of representations $\{\pi_\alpha\}_{\alpha=0}^{\infty}$, that interpolates between the coefficients of the trivial representation at $\alpha = 0$ and the left regular representation at $\alpha = \infty$.

2. Concluding remarks: Norms and strong property (T)

It is natural to ask how the norm $\|\pi\|$ of the representations in Theorem 1 behaves when $\varepsilon \to 0$. The norm of the uniformly bounded representation π induced by renorming of a Hilbert space H_0 via a positive self-adjoint operator T is the number

$$\|\pi\| = \inf\left\{ c \in [1,\infty) \;\middle|\; \begin{array}{l} \pi_{g^{-1}} T \pi_g - cT \text{ is a positive operator} \\ \text{on } H_0 \text{ for every} g \in G \end{array} \right\}.$$

Estimating the above norm does not seem to be an easy task. Since the bottom of the spectrum λ of T converges toward zero as $\varepsilon \to 0$, the right hand side of the estimate $\|\pi\| \leq \frac{\|T\|_{B(\ell_2(G))}}{\lambda}$ tends to infinity and it is natural to expect that the norms of π will blow up to infinity as our coefficients of π approach the trivial representation. For some groups this cannot be improved.

Consider the following strong version of property (T): $H^1(G,\pi) = 0$ *for any uniformly bounded representation π of G on a Hilbert space*. Equivalently,

every affine action with linear part π given by a uniformly bounded representation on a Hilbert space, has a fixed point. This property is possessed by higher rank lattices [Shalom, unpublished], universal lattices [7] and Gromov monsters [8]. As a consequence we have

Proposition 5. *Let G have the above strong property (T) for uniformly bounded representations. Then for any family of uniformly bounded representations π satisfying Theorem 1, $\|\pi\| \to \infty$ as $\varepsilon \to 0$.*

Proof. Assume the contrary. Then for every $\varepsilon > 0$ there exists a uniformly bounded representation $\pi = \pi_\varepsilon$ and vectors v_ε, satisfying the conditions of Theorem 1, with the additional property that $\sup \|\pi_\varepsilon\| \le C$ for some constant $C > 0$.

Choosing a summable sequence of ε we construct a Hilbert space $H = \bigoplus_\varepsilon H_\varepsilon$ and a representation $\rho = \bigoplus \pi_\varepsilon$. By the assumption on the uniform bound on norms of π_ε the representation ρ is uniformly bounded on H. Now construct a cocycle $b_g = \oplus(\pi_\varepsilon)_g v_\varepsilon - v_\varepsilon$. Following the proof of [1] we conclude that b is a proper cocycle, in particular b is not a coboundary. □

References

[1] M. E. B. Bekka, P.-A. Cherix, and A. Valette, *Proper affine isometric actions of amenable groups*, (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser., vol. 227, Cambridge Univ. Press, Cambridge, 1995, pp. 1–4.

[2] M. Cowling, *Unitary and uniformly bounded representations of some simple Lie groups*, Harmonic analysis and group representations, Liguori, Naples, 1982, pp. 49–128.

[3] M. Cowling, *Harmonic analysis on some nilpotent Lie groups (with application to the representation theory of some semisimple Lie groups)*, Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982), Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983, pp. 81–123.

[4] R. G. Douglas and P. W. Nowak, *Hilbert C^*-modules and amenable actions*, Studia Math. **199** (2010), no. 2, 185–197.

[5] E. Guentner, N. Higson, and S. Weinberger, *The Novikov conjecture for linear groups*, Publ. Math. Inst. Hautes Études Sci. **101** (2005), 243–268.

[6] E. Guentner and J. Kaminker, *Exactness and uniform embeddability of discrete groups*, J. London Math. Soc. (2) **70** (2004), no. 3, 703–718.

[7] M. Mimura, *Fixed point properties and second bounded cohomology of universal lattices on Banach spaces*, J. Reine Angew. Math. **653** (2011), 115–134.

[8] A. Naor and L. Silberman, *Poincaré inequalities, embeddings, and wild groups*, Compos. Math. **147** (2011), no. 5, 1546–1572.

[9] P. W. Nowak and G. Yu, *What is ... property A?*, Notices Amer. Math. Soc. **55** (2008), no. 4, 474–475.

[10] P. W. Nowak and G. Yu, *Large scale geometry*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2012.

[11] J.-L. Tu, *Remarks on Yu's "property A" for discrete metric spaces and groups*, Bull. Soc. Math. France **129** (2001), no. 1, 115–139.

[12] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139** (2000), no. 1, 201–240.

Vanderbilt University, Department of Mathematics, 1326 Stevenson Center, Nashville, TN 37240, USA
E-mail address: kate.juschenko@gmail.com

Instytut Matematyczny Polskiej Akademii Nauk, Śniadeckich 8, 00-956 Warszawa, Poland

Institytut Matematyki, Uniwersytet Warszawski, Banacha 2, 02-097 Warszawa, Poland
E-mail address: pnowak@mimuw.edu.pl